Spin polarization analysis at Bloch

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The simplest measurement one can take with the spin detector is scattering asymmetry, for a fixed geometry of the rotator and target magnetization. The normalized scattering asymmetry A in the Ferrum detector is a measured quantity given by:

$$A = \frac{I_{M+} - I_{M-}}{I_{M+} + I_{M-}} \tag{1}$$

with I_{M+} and I_{M+} the scattered intensity as measured by the channeltron, for positive and negative target magnetizations respectively.

In a simplified description we could say that the scattering intensity is given by the scattering cross section:

$$\sigma = \sigma_0(1 + S(\vec{P} \cdot \hat{M})) \tag{2}$$

with S the effective Sherman function (0.29 for the Ferrum detector at Bloch), σ_0 the background spin-independent cross section, $\vec{P}$ the polarization vector of the incoming electron beam and $\hat{M}$ the magnetization direction of the target.

Sanity check: When S=1, we obtain maximum asymmetry with limiting values of $\sigma = 0$ and $\sigma = \sigma_0$ for polarizations parallel and anti-parallel to the target. When S=0, we obtain zero asymmetry with $\sigma = \sigma_0$ for both polarizations.

Px/Py components

We consider first only the $\vec{P_X}$ component, when using a 'rotator +' setting. After the rotator has acted, we have for the geometry at Bloch (see Fig 1):

$$\vec{P_X} \cdot \hat{M}_+ = \frac{1}{\sqrt{2}} ||\vec{P_X}|| \tag{3}$$

$$\vec{P_X} \cdot \hat{M}_- = -\frac{1}{\sqrt{2}} ||\vec{P_X}|| \tag{4}$$

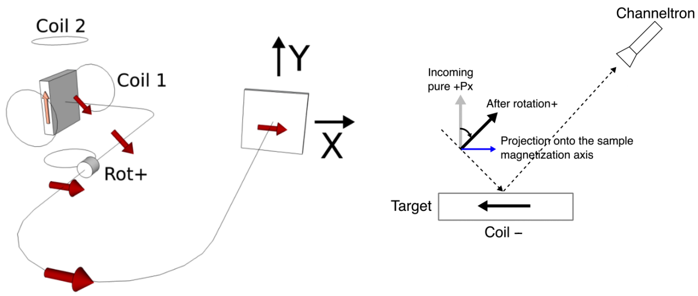

Figure 1: Bloch spin detector geometry, shown measuring a pure $+P_X$ electron beam in the configuration Coil2 minus, Rotator +

Combining these, we can write the $\vec{P_X}$ contribution to A_{R+} as:

$$A_{R+,\vec{P_X}} = \frac{\left(1 + \left(\frac{S||\vec{P_X}||}{\sqrt{2}}\right)\right) - \left(1 + \left(-\frac{S||\vec{P_X}||}{\sqrt{2}}\right)\right)}{\left(1 + \left(\frac{S||\vec{P_X}||}{\sqrt{2}}\right)\right) + \left(1 + \left(-\frac{S||\vec{P_X}||}{\sqrt{2}}\right)\right)} \tag{5}$$

$$= \frac{\frac{2S||\vec{P_X}||}{\sqrt{2}}}{2} \tag{6}$$

$$= \frac{S||\vec{P_X}||}{\sqrt{2}} \tag{7}$$

$$\tag{8}$$

Similar considerations for the $\vec{P_Y}$ component (see Fig 2) give:

$$\vec{P_Y} \cdot \hat{M}_+ = -\frac{1}{\sqrt{2}}||\vec{P_Y}|| \tag{9}$$

$$\vec{P_Y} \cdot \hat{M}_- = \frac{1}{\sqrt{2}}||\vec{P_Y}|| \tag{10}$$

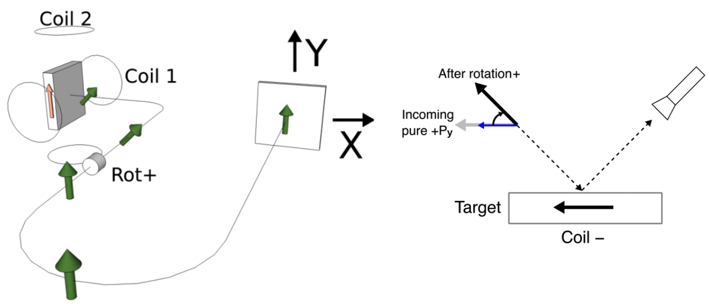

Figure 2: Bloch spin detector geometry, shown measuring a pure $+P_Y$ electron beam in the configuration (coil 2 minus, rotator -)

with a $\vec{P_Y}$ contribution to A_{R+} of:

$$A_{R+,\vec{P_Y}} = \frac{\left(1 + \left(-\frac{S||\vec{P_Y}||}{\sqrt{2}}\right)\right) - \left(1 + \left(\frac{S||\vec{P_Y}||}{\sqrt{2}}\right)\right)}{\left(1 + \left(-\frac{S||\vec{P_Y}||}{\sqrt{2}}\right)\right) + \left(1 + \left(\frac{S||\vec{P_Y}||}{\sqrt{2}}\right)\right)} \tag{11}$$

$$= \frac{-\frac{2S||\vec{P_Y}||}{\sqrt{2}}}{2} \tag{12}$$

$$= \frac{-S||\vec{P_Y}||}{\sqrt{2}} \tag{13}$$

$$\tag{14}$$

Since the $||\vec{P_Z}||$ has no projection onto the Coil 2 magnetization axis for any rotator setting, we now know enough to describe the total A_{R+}:

$$A_{R+} = \frac{S||\vec{P_X}||}{\sqrt{2}} + \frac{-S||\vec{P_Y}||}{\sqrt{2}} \tag{15}$$

Now we go through the same steps for the 'rotator -' setting.
For $\vec{P_X}$ component, after the rotator has acted we have: (see Fig 3):

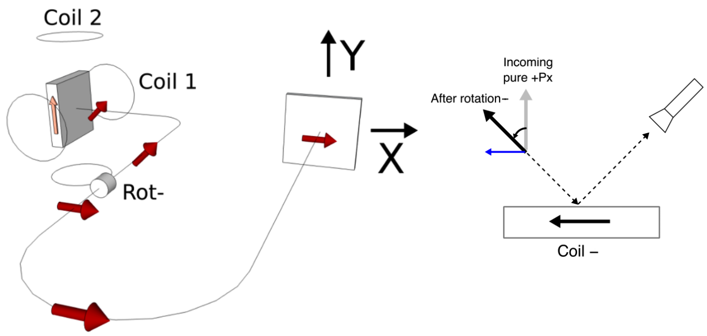

Figure 3: Bloch spin detector geometry, shown measuring a pure $+P_X$ electron beam in the configuration (coil 2 minus, rotator -)

$$\vec{P_X} \cdot \hat{M}_+ = -\frac{1}{\sqrt{2}}||\vec{P_X}|| \tag{16}$$

$$\vec{P_X} \cdot \hat{M}_- = \frac{1}{\sqrt{2}}||\vec{P_X}|| \tag{17}$$

$$A_{R-,\vec{P_X}} = \frac{\left(1 + \left(\frac{-S||\vec{P_X}||}{\sqrt{2}}\right)\right) - \left(1 + \left(\frac{S||\vec{P_X}||}{\sqrt{2}}\right)\right)}{\left(1 + \left(\frac{-S||\vec{P_X}||}{\sqrt{2}}\right)\right) + \left(1 + \left(\frac{S||\vec{P_X}||}{\sqrt{2}}\right)\right)} \tag{18}$$

$$= \frac{-\frac{2S||\vec{P_X}||}{\sqrt{2}}}{2} \tag{19}$$

$$= \frac{-S||\vec{P_X}||}{\sqrt{2}} \tag{20}$$

$$\tag{21}$$

While for the $\vec{P_Y}$ component, after the rotator has acted we have: (see Fig 4):

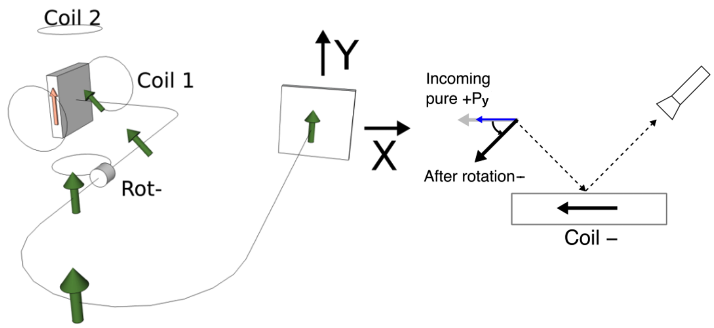

Figure 4: Bloch spin detector geometry, shown measuring a pure $+P_Y$ electron beam in the configuration (coil 2 minus, rotator -)

$$\vec{P_Y} \cdot \hat{M}_+ = -\frac{1}{\sqrt{2}}||\vec{P_Y}|| \tag{22}$$

$$\vec{P_Y} \cdot \hat{M}_- = \frac{1}{\sqrt{2}}||\vec{P_Y}|| \tag{23}$$

$$A_{R-,\vec{P_Y}} = \frac{\left(1+\left(\frac{-S||\vec{P_Y}||}{\sqrt{2}}\right)\right)-\left(1+\left(\frac{S||\vec{P_Y}||}{\sqrt{2}}\right)\right)}{\left(1+\left(\frac{-S||\vec{P_Y}||}{\sqrt{2}}\right)\right)+\left(1+\left(\frac{S||\vec{P_Y}||}{\sqrt{2}}\right)\right)} \tag{24}$$

$$= \frac{-\frac{2S||\vec{P_Y}||}{\sqrt{2}}}{2} \tag{25}$$

$$= \frac{-S||\vec{P_Y}||}{\sqrt{2}} \tag{26}$$

$$\tag{27}$$

with the total A_{R-} given by:

$$A_{R-} = \frac{-S||\vec{P_X}||}{\sqrt{2}} + \frac{-S||\vec{P_Y}||}{\sqrt{2}} \tag{28}$$

At this point, a key insight is that we can disentangle the $\vec{P_X}$ and $\vec{P_Y}$ components by taking the sum and difference of these two asymmetries:

$$A_{R+} - A_{R-} = \left(\frac{S||\vec{P_X}||}{\sqrt{2}} + \frac{-S||\vec{P_Y}||}{\sqrt{2}}\right) - \left(\frac{-S||\vec{P_X}||}{\sqrt{2}} + \frac{-S||\vec{P_Y}||}{\sqrt{2}}\right) \tag{29}$$

$$= \frac{2S||\vec{P_X}||}{\sqrt{2}} \tag{30}$$

$$= \sqrt{2}S||\vec{P_X}|| \tag{31}$$

$$\boxed{||\vec{P_X}|| = \frac{1}{\sqrt{2}S}\Big(A_{R+} - A_{R-}\Big)} \tag{32}$$

$$A_{R+} + A_{R-} = \left(\frac{S||\vec{P_X}||}{\sqrt{2}} + \frac{-S||\vec{P_Y}||}{\sqrt{2}}\right) + \left(\frac{-S||\vec{P_X}||}{\sqrt{2}} + \frac{-S||\vec{P_Y}||}{\sqrt{2}}\right) \tag{33}$$

$$= \frac{-2S||\vec{P_Y}||}{\sqrt{2}} \tag{34}$$

$$= -\sqrt{2}S||\vec{P_Y}|| \tag{35}$$

$$\boxed{||\vec{P_Y}|| = -\frac{1}{\sqrt{2}S}\Big(A_{R+} + A_{R-}\Big)} \tag{36}$$

Pz component

Obtaining $\vec{P_Z}$ is considerably simpler, since this component arrives to the target aligned to the 'coil 1' magnetization axis and the rotator has no influence.

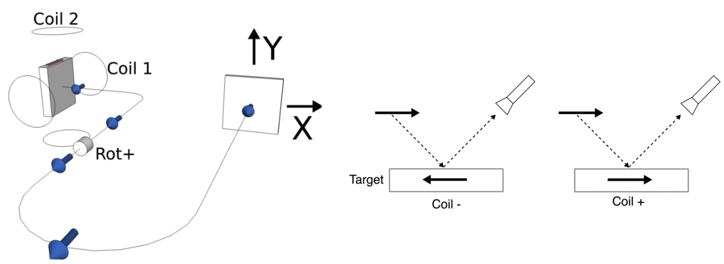

Figure 5: Bloch spin detector geometry, shown measuring a pure $+P_Z$ electron beam in the configuration (coil 1 minus)

$$\vec{P_Z} \cdot \hat{M}_+ = ||\vec{P_Z}|| \tag{37}$$

$$\vec{P_Z} \cdot \hat{M}_- = -||\vec{P_Z}|| \tag{38}$$

$$A = \frac{\Big(1 + \Big(S||\vec{P_Z}||\Big)\Big) - \Big(1 + \Big(-S||\vec{P_Z}||\Big)\Big)}{\Big(1 + \Big(S||\vec{P_Z}||\Big)\Big) + \Big(1 + \Big(-S||\vec{P_Z}||\Big)\Big)} \tag{39}$$

$$= \frac{2S||\vec{P_Z}||}{2} \tag{40}$$

$$= S||\vec{P_Z}|| \tag{41}$$

$$\tag{42}$$

$$\boxed{||\vec{P_Z}|| = A/S} \tag{43}$$

What if the rotator orientation were opposite?

This would require swapping the A_{R+} and A_{R-} terms. Since the $||\vec{P_Y}||$ component is given by the asymmetry **sum**, it would be unaffected. If the rotator action were the opposite to our assumption, **the effect would be an inversion of all $||\vec{P_X}||$ components**.

What if the target magnetization were opposite?

This would invert the sign of all $\vec{P} \cdot \hat{M}$ terms. If the magnetization direction were the opposite to our assumption, **the effect would be an inversion of all 3 components**.

Component intensities

It is common to resolve the total, spin integrated intensity I_{total} into two spin-resolved 'component intensities' or 'partial intensities'. For spin polarization along the x axis, the component intensities are given by:

$$C_+ = \frac{I_{total}}{2}\left(1 + P_x\right) \tag{44}$$

$$C_- = \frac{I_{total}}{2}\left(1 - P_x\right) \tag{45}$$

It is straightforward to confirm that $C_+ + C_- = I_{total}$ as expected, and that for limiting values of $P_x = \pm 1$ we obtain $C_\pm = I_{total}$ and $C_\mp = 0$.

Error bars

Here we will consider only statistical errors from the channeltron signal. It is important to recognize that many other, potentially much larger sources of error can be introduced by independent factors such as poor beam alignment.

For simplicity we begin with the Pz component, for which only two measurements are needed. The statistical error associated with a measurement from a channeltron is generally considered to be governed by Poisson statistics, such that:

$$\Delta N = \sqrt{N} \tag{46}$$

We can use Eqn 1 from earlier to propagate the error of the two channeltron measurements (±target magnetization) into the error in the asymmetry. Following the derivation described in Kessler p242:

$$\Delta A^2 = \left(\frac{\partial A}{\partial N_{M+}}\right)^2 (\Delta N_{M+})^2 + \left(\frac{\partial A}{\partial N_{M-}}\right)^2 (\Delta N_{M-})^2 \tag{47}$$

$$= \left(\frac{2N_{M-}}{(N_{M+}+N_{M-})^2}\right)^2 N_{M+} + \left(\frac{-2N_{M+}}{(N_{M+}+N_{M-})^2}\right)^2 N_{M-} \tag{48}$$

Letting $N = N_{M+} + N_{M-}$, we obtain:

$$\Delta A^2 = \frac{4N_{M+}N_{M-}}{N^3} \tag{49}$$

We're now able to do a simplification trick. Since we know from Eqn 43 that $A = SP_Z$, we can construct the term:

$$1 - S^2 P_Z^2 = 1 - A^2 \tag{50}$$

$$= 1 - \left(\frac{N_{M+} - N_{M-}}{N_{M+} + N_{M-}}\right)^2 \tag{51}$$

$$= \frac{4N_{M+}N_{M-}}{N^2} \tag{52}$$

This allows us to simplify Eqn 49 to:

$$\Delta A = \sqrt{\frac{1 - S^2 P_Z^2}{N}} \tag{53}$$

Combining Eqns 43 and 53, we have:

$$\Delta P_Z = \frac{\Delta A}{S} \tag{54}$$

$$= \frac{1}{S}\sqrt{\frac{1 - S^2 P_Z^2}{N}} \tag{55}$$

$$= \sqrt{\frac{1}{N}\left(\frac{1}{S^2} - P_Z^2\right)} \tag{56}$$

$$\tag{57}$$

One could interpret the term $\frac{1}{S^2} - P_Z^2$ as 'how much information does each detection convey?'. This depends on both the Sherman function of the detector and the polarization of the beam. In the limit S=1, P=±1, there is zero uncertainty and a single scattered electron count would be sufficient to know the state of the beam. Here in the real world we're seldom close to these limits and more detections are needed ($\frac{1}{N}$ term) in order to resolve the signal from the noise. For contemporary spin detectors the $\frac{1}{S^2}$ term (≈12 if S=0.29) dominates the P_Z^2 term (maximum 1) and a common approximation is to ignore the P_Z^2 term:

$$\boxed{\Delta P_Z = \sqrt{\frac{1}{NS^2}}} \tag{58}$$

It follows from Eqn 54 that:

$$\boxed{\Delta A_Z = \sqrt{\frac{1}{N}}} \tag{59}$$

Here the Sherman function has dropped out, but this seems reasonable: it changes only the *magnitude* of the asymmetry, not our *certainty* about it.

For the component intensities $C_\pm$ defined in Eqns 44 and 45, we will have:

$$\Delta C_\pm^2 = \left(\frac{\partial C_\pm}{\partial P}\right)^2 (\Delta P)^2 \tag{60}$$

$$\Delta C_\pm = \frac{N}{2}(\Delta P) \tag{61}$$

$$= \frac{N}{2}\sqrt{\frac{1}{NS^2}} \tag{62}$$

$$\tag{63}$$

But what of the Px/Pz components, which in our somewhat unusual (Ferrum + rotator) setup involves combining the results of 4 channeltron measurements? (±target magnetization, ±rotator). Is it valid to simply sum the four intensities and reuse equation 59?

An analogous derivation to the one above would begin with propagating the errors for the sum and for the difference of the asymmetries with the different rotator settings, i.e. starting from

$$P_X = \frac{\sqrt{2}}{2S} A_{DIFF} \tag{64}$$

$$P_Y = -\frac{\sqrt{2}}{2S} A_{SUM} \tag{65}$$

where for notational convenience we are taking:

$$A_{DIFF} = A_{R+} - A_{R-} \tag{66}$$

$$A_{SUM} = A_{R+} + A_{R-} \tag{67}$$

and will also define the channeltron counts for the four different configurations as:

$$a = N_{\substack{M+\\R+}} b = N_{\substack{M-\\R+}} c = N_{\substack{M+\\R-}} d = N_{\substack{M-\\R-}} \tag{68}$$

$$\Delta(A_{DIFF})^2 = \left(\frac{\partial A_{DIFF}}{\partial a}\right)^2 (\Delta a)^2 + \left(\frac{\partial A_{DIFF}}{\partial b}\right)^2 (\Delta b)^2 + \left(\frac{\partial A_{DIFF}}{\partial c}\right)^2 (\Delta c)^2 + \left(\frac{\partial A_{DIFF}}{\partial d}\right)^2 (\Delta d)^2 \tag{69}$$

$$= \left(\frac{4ab}{(a+b)^3} + \frac{4cd}{(c+d)^3}\right) \tag{70}$$

$$\Delta(A_{SUM})^2 = \left(\frac{\partial A_{SUM}}{\partial a}\right)^2 (\Delta a)^2 + \left(\frac{\partial A_{SUM}}{\partial b}\right)^2 (\Delta b)^2 + \left(\frac{\partial A_{SUM}}{\partial c}\right)^2 (\Delta c)^2 + \left(\frac{\partial A_{SUM}}{\partial d}\right)^2 (\Delta d)^2 \tag{71}$$

$$= \frac{4ab}{(a+b)^3} + \frac{4cd}{(c+d)^3} \tag{72}$$

As expected, the error doesn't depend on whether we add or subtract the terms! The error of the individual asymmetry terms A_{R+} and A_{R-} may differ, but the sum and difference of them should should have the same error. So for simplicity, we'll only refer to A_{SUM} in what follows.

A substitution trick in the spirit of what we did for Pz would suggest the terms:

$$1 - A_{R+}^2 = \frac{4ab}{(a+b)^2} = 1 - \frac{S^2}{2}(P_X - P_Y)^2 \tag{73}$$

and:

$$1 - A_{R-}^2 = \frac{4cd}{(c+d)^2} = 1 - \frac{S^2}{2}(-P_X - P_Y)^2 \tag{74}$$

Substituting this into our expression for $(\Delta A_{SUM})^2$, we get:

$$(\Delta A_{SUM})^2 = \frac{1}{(a+b)}\frac{4ab}{(a+b)^2} + \frac{1}{(c+d)}\frac{4cd}{(c+d)^2} \tag{75}$$

$$= \frac{1}{(a+b)}\left(1 - \frac{S^2}{2}(P_X - P_Y)^2\right) + \frac{1}{(c+d)}\left(1 - \frac{S^2}{2}(-P_X - P_Y)^2\right) \tag{76}$$

$$\tag{77}$$

Now since $P_Y = -\frac{1}{\sqrt{2}S}A_{SUM}$, we can say:

$$(\Delta P_Y)^2 = \frac{1}{2S^2}(\Delta A_{SUM})^2 \tag{78}$$

$$= \frac{1}{2S^2}\left[\frac{1}{(a+b)}\left(1-\frac{S^2}{2}(P_X-P_Y)^2\right)+\frac{1}{(c+d)}\left(1-\frac{S^2}{2}(-P_X-P_Y)^2\right)\right] \tag{79}$$

$$= \frac{1}{2S^2(a+b)} - \frac{(P_X-P_Y)^2}{4(a+b)} + \frac{1}{2S^2(c+d)} - \frac{(-P_X-P_Y)^2}{4(c+d)} \tag{80}$$

$$= \frac{1}{2(a+b)}\left(\frac{1}{S^2}-\frac{(P_X-P_Y)^2}{2}\right)+\frac{1}{2(c+d)}\left(\frac{1}{S^2}-\frac{(-P_X-P_Y)^2}{2}\right) \tag{81}$$

$$\tag{82}$$

We can now make the same approximation as was used for ΔP_Z, noting that the $\frac{1}{S^2}$ terms (≈12) will dominate the $\frac{1}{2}(\pm P_X - P_Y)^2$ term (maximum 2). This reduces our expression to:

$$(\Delta P_Y)^2 = \frac{1}{2}\left(\frac{1}{S^2(a+b)}+\frac{1}{S^2(c+d)}\right) \tag{83}$$

When deriving the error term for Pz we used N to denote the sum of counts for the two configurations in the measurement. We're now making four measurements in a Px/Py set,and since typically the countrate is similar for the two rotator settings, we can say $(a+b) = (c+d) = N$. Substituting that in gives:

$$\Delta P_Y = \sqrt{\frac{1}{2}\left(\frac{2}{S^2N}\right)} \tag{84}$$

$$= \sqrt{\frac{1}{NS^2}} \tag{85}$$

$$\tag{86}$$

i.e. for the same countrate and the same total measurement time per configuration, the statistical error on the $P_{X,Y}$ measurements is the same as for a P_Z measurement.

But be careful about the definition of N here. For Pz this is the total number of counts across the two target magnetization measurement sets. For Px/Py it is **half** the total number of counts across all four of the measurements sets. $N = (a+b) = (c+d)$, $N \neq (a+b+c+d)$